# Math 6261 2023-02-24.

## Chap. 3 The Central limit theorem.

The central limit theorem (CLT) is one of the greatest result of Mathematics.

### § 3.1 Weak convergence.

Let $\text{Prob}(\mathbb{R})$ denote the space of Borel prob. measures on $\mathbb{R}$, and $C_b(\mathbb{R})$ for the space of bounded continuous functions on $\mathbb{R}$.

**Def.** Let $\mu_n \in \text{Prob}(\mathbb{R})$, $n=1,2,\cdots$, and $\mu \in \text{Prob}(\mathbb{R})$.
Say that $\mu_n$ converges weakly to $\mu$ if

$$\int h \, d\mu_n \to \int h \, d\mu \quad \text{for all } h \in C_b(\mathbb{R})$$

and then write

$$\mu_n \xrightarrow{w} \mu.$$

Notice that the elements of $\text{Prob}(\mathbb{R})$ correspond to distribution functions via

$$\mu \leftrightarrow F, \qquad \text{where} \quad F(x) = \mu(-\infty, x].$$

This will be proved in Thm 3.3.

**Def.** Weak convergence of distribution functions is defined as follows:

$$F_n \xrightarrow{w} F \quad \text{if and only if} \quad \mu_n \xrightarrow{w} \mu.$$

We are generally interested in the case when $F_n = F_{X_n}$, that is when $F_n$ is the distribution function of $X_n$.

**Def:** Let $X_n$, $n \geq 1$, and $X$ are r.v.'s on the same triple $(\Omega, \mathcal{F}, P)$.

Write $X_n \xrightarrow{w} X$ if $F_{X_n} \xrightarrow{w} F_X$.

Equivalenty

$X_n \xrightarrow{w} X$ if $E\, h(X_n) \to E\, h(X)$ for all $h \in C_b(\mathbb{R})$.

**Remark:** (1) $X_n \to X$ a.s. $\Rightarrow$ $X_n \xrightarrow{w} X$ (by DCT).

(2) $X_n \to X$ in probabitity $\Rightarrow$ $X_n \xrightarrow{w} X$.

(3) $X_n \xrightarrow{w} X$ $\not\Rightarrow$ $X_n \to X$ in probability.

Remark: $F_n \xrightarrow{w} F$ does not imply that $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$; see Example 3.1.

Example 3.1 Let $\mu_n$ be the unit mass at $\frac{1}{n}$, $n = 1, 2, \cdots$ and $\mu$ be the unit mass at $0$.

Then for each $h \in C_b(\mathbb{R})$

$$\int h \, d\mu_n = h\left(\frac{1}{n}\right) \longrightarrow h(0) = \int h \, d\mu.$$

Hence $\mu_n \xrightarrow{w} h$.

However, $F_n(0) = \mu_n(-\infty, 0] = 0$, but $F(0) = \mu(-\infty, 0] = 1$.

Hence $F_n(0) \not\to F(0)$.

Prop. 3.2. Let $(F_n)$ be a sequence of distribution functions (DFs) on $\mathbb{R}$, and let $F$ be a DF on $\mathbb{R}$. Then

$F_n \xrightarrow{w} F$ if and only if

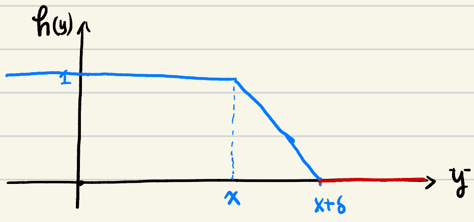
$$\lim_{n\to\infty} F_n(x) = F(x)$$

for every non-atom (that is, every continuity pt) $x$ of $F$.

Pf. "Only if" part. Suppose that $F_n \xrightarrow{w} F$.

Let $x \in \mathbb{R}$ and $\delta > 0$. Define $h \in C_b(\mathbb{R})$ by

$$h(y) = \begin{cases} 1 & \text{if } y \leq x \\ 1 - \frac{1}{\delta}(y-x) & \text{if } x < y < x+\delta \\ 0 & \text{if } y \geq x+\delta \end{cases}$$




Then $F_n(x) = \mu_n(-\infty, x] \leq \int h \, d\mu_n \to \int h \, d\mu$
$\leq \mu(-\infty, x+\delta]$
$= F(x+\delta)$.

Hence $$\overline{\lim_{n\to\infty}} F_n(x) \leq F(x+\delta).$$

Letting $\delta \downarrow 0$ and using the right continuity of $F$, we obtain

$$\overline{\lim_{n\to\infty}} F_n(x) \le F(x).$$

Similarly, working with $y \mapsto h(y+\delta)$. set $h^*(y) = h(y+\delta)$.

$$\underline{\lim_{n\to\infty}} F_n(x) \ge \underline{\lim_{n\to\infty}} \int h^* \, d\mu_n = \int h^* \, d\mu$$
$$\ge F(x-\delta)$$

Letting $\delta \downarrow 0$ gives

$$\underline{\lim_{n\to\infty}} F_n(x) \ge F(x-).$$

This proves the "only if" part.

The "if" part of the proposition is based on the following representation result.

Thm 3.3 ( Skorokhod representation)

Suppose that $(F_n)$ is a sequence of DFs on $\mathbb{R}$ and $F$ a DF on $\mathbb{R}$ such that

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{at every cty pt } x \text{ of } F.$$

Then $\exists$ a prob. space $(\Omega, \mathcal{F}, P)$ which carries a sequence of r.v.'s $X_n$, and a r.v. $X$ such that

$$F_n = F_{X_n}, \quad F = F_X \quad \text{and}$$

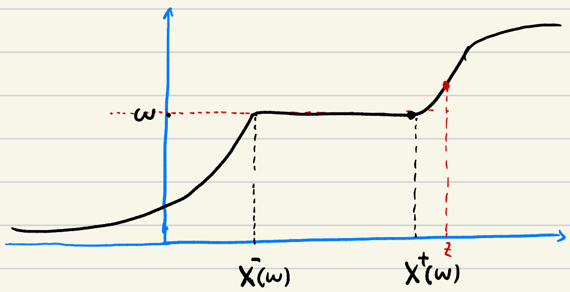
$X_n \to X$ a.s.

Pf. Define $(\Omega, \mathcal{F}, P) = \left( [0,1], \mathcal{B}([0,1]), \text{Leb} \right)$.

Set

$$X^+(\omega) = \inf\{ z : F(z) > \omega \}$$

$$X^-(\omega) = \inf\{ z : F(z) \geq \omega \}.$$




Define $X_n^+$ and $X_n^-$ similarly. Below we show that

① $P(X^- \leq z) = F(z)$.

② $P(X^+ \leq z) = F(z)$.

③ $P(X^+ \neq X^-) = 0$.

We first prove ①. Let $\omega \in [0,1]$. By the definition of $X^-(\omega)$,

$\exists\ z_n \downarrow X^-(\omega)$ such that $F(z_n) \geq \omega$.

By the right continuity of F,

$$F(\bar{X}(w)) = \lim_{n\to\infty} F(z_n) \geq w.$$

So by the non-decreasing property of F,

$$\bar{X}(w) \leq z \Rightarrow F(\bar{X}(w)) \leq F(z)$$

$$\Rightarrow w \leq F(z).$$

Thus $(\bar{X} \leq z) \subset [0, F(z)]$

so

$$P(\bar{X} \leq z) \leq F(z).$$

On the other hand, by the definition of $\bar{X}$,

$$z < \bar{X}(w) \Rightarrow F(z) < w.$$

i.e. $(\bar{X} > z) \subset \{w : F(z) < w \leq 1\}$.

Hence $P(\bar{X} > z) \leq 1 - F(z)$, which implies that

$$P(\bar{X} \leq z) = 1 - P(\bar{X} > z) \geq 1 - (1 - F(z)) \geq F(z)$$

Now we obtain $P(\bar{X} \leq z) = F(z)$.

Next we show that

$$P(X^+ \leq z) = F(z).$$

Clearly $P(X^+ \leq z) \leq P(X^- \leq z) = F(z)$.

Notice that by the definition of $X^+$,

$$z < X^+(w) \Rightarrow F(z) \leq w$$

So $\quad (X^+ > z) \subset \{w : F(z) \leq w \leq 1\}$

Thus $P(X^+ > z) \leq 1 - F(z) \Rightarrow P(X^+ \leq z) \geq F(z)$.

This establishes $P(X^+ \leq z) = F(z)$.

Now we show that

$$P(X^+ \neq X^-) = 0.$$

Clearly $(X^+ \neq X^-) = \bigcup_{c \in \mathbb{Q}} (X^- \leq c < X^+)$

Notice that $P(X^- \leq c < X^+) = P(X^- \leq c) - P(X^+ \leq c)$

$$= F(c) - F(c) = 0.$$

It follows that $P(X^+ \neq X^-) = 0$.

Finally we prove that $X_n \to X$ a.s.

Fix $\omega$. Let $z$ be a non-atom of $F$ with $z > X^+(\omega)$.

Then $F(z) > \omega$. Since $\lim_{n\to\infty} F_n(z) = F(z)$, we have

$$F_n(z) > \omega \quad \text{when } n \text{ is large enough.}$$

By definition, $X_n^+(\omega) \le z$ for large $n$.

Hence $$\overline{\lim_{n\to\infty}}\, X_n^+(\omega) \le z.$$

Since non-atoms are dense, we obtain

$$\overline{\lim_{n\to\infty}}\, X_n^+(\omega) \le X^+(\omega).$$

We can prove $\underline{\lim}_{n\to\infty} X_n^+(\omega) \ge X^-(\omega)$ in a similar way. Indeed let $z$ be a non-atom of $F$ with $z < X^-(\omega)$. Then by definition of $X^-(\omega)$, $F(z) < \omega$. Hence for large $n$,

$$F_n(z) < \omega.$$

It follows that $z \le X_n^-(\omega)$ for large $n$.

So $\underline{\lim}_{n\to\infty} X_n^-(\omega) \ge z$

Again since non-atoms are dense, we have $\underline{\lim}_{n\to\infty} X_n^-(\omega) \ge X^-(\omega)$.

However since $P(X^+ = X^-) = 1$, it follows that

$$X_n \to X \quad \text{a.s.}$$

$\square$

The below example shows that the space $\mathrm{Prob}(\bar{\mathbb{R}})$, endowed with the weak topology, is not compact.

Example 3.4. Let $\mu_n$ be the unit mass at $n$. Then no subsequence of $(\mu_n)$ converges weakly in $\mathrm{Prob}(\mathbb{R})$.

To see it, suppose on the contrary that

$$\mu_{n_k} \to \mu$$

for some $\mu \in \mathrm{Prob}(\mathbb{R})$. Take $h \in C_b(\mathbb{R})$ such that $\mathrm{supp}\, h$ is compact, $h \geq 0$, $\int h \, d\mu > 0$.

Then $\int h \, d\mu_{n_k} = h(n) \to 0$ as $n \to +\infty$.

So $\lim_{k\to\infty} \int h \, d\mu_{n_k} \neq \int h \, d\mu$.

Let $\bar{\mathbb{R}} = [-\infty, \infty]$ endowed with the usual topology. Then $\bar{\mathbb{R}}$ is a compact metric space.

By the Riesz representation Thm, the dual space $C(\bar{\mathbb{R}})^*$ of $C(\bar{\mathbb{R}})$ is the space of all signed measures. The weak* topology of $C(\bar{\mathbb{R}})^*$ is metrizable, and under this topology the unit ball of $C(\bar{\mathbb{R}})^*$ is compact, and contains $\text{Prob}(\bar{\mathbb{R}})$ as a closed subset.

The weak* topology of $\text{Prob}(\bar{\mathbb{R}})$ is exactly the weak topology.

Hence

$\text{Prob}(\bar{\mathbb{R}})$ is a compact metrizable space under our probabilists' weak topology.

(Helley-Bray)

**Prop 3.5.** Let $(F_n)$ be any sequence of DFs on $\mathbb{R}$. Then $\exists$ a right continuous non-decreasing function $F$ on $\mathbb{R}$ such that

$0 \le F \le 1$ and a subsequence $(n_i)$ such that

$$\lim_{i\to\infty} F_{n_i}(x) = F(x)$$

at every continuity point $x$ of $F$.

Pf. Every $F_n$ corresponds to a unique $\mu_n \in \text{Prob}(\mathbb{R})$ such that

$$F_n(z) = \mu_n(-\infty, z] \quad \text{for } z \in \mathbb{R}.$$

But each $\mu_n$ can be viewed as an element in $\text{Prob}(\bar{\mathbb{R}})$.

Since $\text{Prob}(\bar{\mathbb{R}})$ is compact, so $\exists$ a subsequence $(n_i)$ and $\mu \in \text{Prob}(\bar{\mathbb{R}})$ such that

$$\mu_{n_i} \xrightarrow{w} \bar{\mu}$$

Let $F(z) = \mu([-\infty, z])$. Then $F$ is right cts and non-decreasing

$F_{n_k}(z) \to F(z)$ at continuity point of $F$. $\blacksquare$

Remark: • The function $F$ in Prop 3.5 may not be a DF, since it may happen that $\lim_{x \to -\infty} F(x) \neq 0$ or $\lim_{x \to +\infty} F(x) \neq 1$.

Equivalently, $\bar{\mu}\{-\infty, +\infty\} > 0$.